The Notorious 15 Box Puzzle

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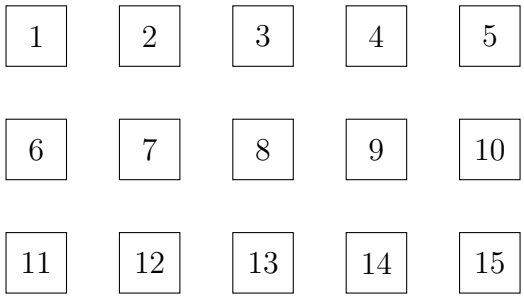
Intro

In all my years of puzzling I have not seen such a simple question spark so much controversy. This lovely intuition-bending puzzle involves 15 boxes and 2 contestants. Who is more likely to win? Or is it a fair game?

Since this question only has 3 possible answers, it can be considered an intuition puzzle. You, the puzzler, are encouraged to register a concrete guess before thinking too rigorously about the problem. Then try to justify (or disprove) yourself for the best fun. [1]

Riddle

The situation is simple. There are 15 boxes arranged in a 3×5 format as follows.



The contest organizers have laid out these 15 boxes, and have selected two of them uniformly at random to contain goats.

The two contestants, Alice and Bob, will open boxes in parallel to try and find a goat first! Both players begin at the top-left box, and open it together in the first round. Then, both players proceed by opening 1 box per round in a predetermined order as follows.

[1] I saw this riddle on Gil Kalai's blog: Combinatorics and More.

Alice opens boxes in "row-major" order, so her box-opening sequence is simply $1, 2, 3, \ldots, 14, 15$. Bob goes by columns, from top to bottom, and then left to right. So his box-opening sequence is $1, 6, 11, 2, 7, 12, 3 \ldots, 10, 15$.

If Alice and Bob open a box at the same time and it contains a goat, then the game is a tie. If Alice and Bob open two different boxes in the same round, and they both contains a goat, then it is also a tie. In any other case, whoever opens a box with a goat first wins the grand prize: another goat.

Riddle Question: Who is more likely to win the game? Or are Alice and Bob equally likely to win?

Hints

Hint 1: Suppose that, instead of a 3×5 layout, the organizers arranged nine boxes in a 3×3 square. In this case, you should be able to prove that Alice and Bob are equally likely to win the game. Hence, it may be helpful to take the asymmetry in this puzzle to the extreme case. What if the layout was 2×100?

Hint 2: It may help to consider the case of 1 prize. If only one box is selected to have a goat in it, and Alice and Bob stick with their box-opening sequences as above, are they equally likely to win the game? Does this hold for any pair of deterministic box-ordering sequence by Alice and Bob? Be careful here.

Hint 3: Mark a box with an A if Alice reaches the box before Bob, and with a B if Bob reaches the box before Alice. The tied boxes can be ignored since they result in a tie. How many A boxes are there compared with B boxes? This should answer the question posed in Hint 2.

Now, notice that if both goats are placed in A boxes, then Alice will certainly win. Similarly, if both goats land in B boxes, then Bob will win. The only tricky case is when one goat is hiding in an A box, and another is in a B box. In this case, it all depends on who reaches their goat box first.

Hint 4: Label the A and B boxes with the time that Alice and Bob (respectively) reach them. Does one sequence dominate the other? This should provide the answer you seek.

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